Quantum bits

Quantum computing

G. Chênevert

Jan. 8, 2021

Quantum bits

Quantum systems

Dirac formalism

Quantum bits

Recall: Wave function

A quantum system can be described by a (complex-valued) **wave function** $\Psi(\mathbf{x}, t)$

satisfying **Schrödinger's equation**:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V \Psi$$

where

- $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator,
- $V(\mathbf{x}, t)$ the potential function representing the environment.

Stationary states

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V\,\Psi$$

Let's assume that the potential $V = V(\mathbf{x})$ is independent of t and look for *separable solutions* of the form

$$\Psi(\mathbf{x}, t) = \chi(t)\,\phi(\mathbf{x}).$$

The equation becomes: $$i\hbar \frac{\partial \chi}{\partial t}\,\phi = \chi\left(-\frac{\hbar^2}{2m}\Delta\phi + V\phi\right)$$

or

$$\frac{i\hbar}{\chi}\frac{\partial \chi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\Delta\phi}{\phi} + V.$$

Separable solutions

$$\frac{i\hbar}{\chi}\frac{\partial\chi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\Delta\phi}{\phi} + V = \text{constant} =: E$$

reduces to

$$\begin{cases} \dfrac{\partial\chi}{\partial t} = -\dfrac{iE}{\hbar}\chi \\ -\dfrac{\hbar^2}{2m}\Delta\phi + V\phi = E\phi \end{cases}$$

$$\implies \begin{cases} \chi(t) = A\,e^{-\frac{iE}{\hbar}t} \quad \text{and} \\ \widehat{H}\,\phi = E\,\phi \qquad \text{where} \qquad \widehat{H} = -\dfrac{\hbar^2}{2m}\Delta + V \end{cases}$$

Quantization

Given boundary conditions on $\phi(\mathbf{x})$, the reduced Hamiltonian operator $\widehat{H}$ only has countably many (real) eigenvalues:

$$E_0 \leq E_1 \leq E_2 \leq \cdots \leq E_n \leq \cdots,$$

corresponding to countably many eigenfunctions:

$$\phi_0, \quad \phi_1, \quad \phi_2, \quad \ldots \quad \phi_n, \quad \ldots$$

hence we get countably many separable solutions

$$\Psi_n(\mathbf{x}, t) = A_n \, e^{-\frac{iE_n}{\hbar} t} \, \phi_n(\mathbf{x}).$$

Quantum states

In general, the state of a quantum system can be written as a linear combination

$$\Psi(\mathbf{x}, t) = \sum_n A_n \, e^{-i \frac{E_n}{\hbar} t} \, \phi_n(\mathbf{x})$$

where the ϕ_n are eigenfunctions for the reduced Hamiltonian operator:

$$\widehat{H} \, \phi_n = E_n \, \phi_n.$$

These eigenstates are orthogonal with respect to the Hermitian product

$$\langle \phi \, | \, \psi \rangle = \int \phi(\mathbf{x})^* \, \psi(\mathbf{x}) \, \mathrm{d}\mathbf{x}$$

Quantum bits

Quantum systems

Dirac formalism

Quantum bits

Braket notation

The instantaneous states $\phi(\mathbf{x}) = \Psi(\mathbf{x}, t_0)$ form a vector space $\mathcal{V}$ spanned by the ϕ_n:

$$\phi(\mathbf{x}) = \sum_n \alpha_n \, \phi_n(\mathbf{x}) \qquad \text{with} \qquad \alpha_n \in \mathbb{C}.$$

Hermitian product: if the ϕ_n are **normalized** ($\|\phi_n\| = \sqrt{\langle \phi_n | \phi_n \rangle} = 1$) then for

$$\phi = \sum_n \alpha_n \, \phi_n, \qquad \psi = \sum_n \beta_n \, \phi_n,$$

we have

$$\langle \phi \,|\, \psi \rangle = \sum_n \alpha_n^* \, \beta_n = \begin{bmatrix} \alpha_0 & \alpha_1 & \dots & \end{bmatrix}^* \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \end{bmatrix} = |\phi\rangle^\dagger \, |\psi\rangle$$

Measurement

When we measure a **mixed state**

$$|\phi\rangle = \sum_n \alpha_n |\phi_n\rangle \in \mathcal{V} \setminus \{\mathbf{0}\} :$$

it gets projected on the **pure state** $|\phi_n\rangle$ with energy E_n with probability

$$\mathbb{P}\big[\,\mathcal{M}|\phi\rangle = |\phi_n\rangle\,\big] = \frac{|\langle \phi \mid \phi_n \rangle|^2}{\|\phi\|^2} = \frac{|\alpha_n|^2}{\|\phi\|^2}.$$

If $|\phi\rangle$ is normalized, this is just

$$\mathbb{P}\big[\,\mathcal{M}|\phi\rangle = |\phi_n\rangle\,\big] = |\langle \phi \mid \phi_n \rangle|^2 = |\alpha_n|^2.$$

Exercise

We measure the mixed quantum state

$$|\phi\rangle = |\phi_0\rangle + (3+4i)|\phi_1\rangle + 7|\phi_2\rangle + 5i|\phi_3\rangle.$$

What to we expect to see ?

Answer:

$$\mathbb{P}\big[\,\mathcal{M}|\phi\rangle = |\phi_n\rangle\,\big] = \begin{cases} 1\% & n = 0 \\ 25\% & n = 1 \\ 49\% & n = 2 \\ 25\% & n = 3 \end{cases}$$

Equivalence

When two states are proportional: $|\phi\rangle = \alpha\,|\psi\rangle$ $(\alpha \neq 0)$ then

$$\mathbb{P}\big[\,\mathcal{M}|\phi\rangle = |\phi_n\rangle\,\big] = \frac{|\langle\phi\,|\,\phi_n\rangle|^2}{\|\phi\|^2} = \frac{|\alpha|^2\,|\langle\psi\,|\,\phi_n\rangle|^2}{|\alpha|^2\,\|\psi\|^2} = \mathbb{P}\big[\,\mathcal{M}|\psi\rangle = |\phi_n\rangle\,\big]$$

Thus $|\phi\rangle$ and $|\psi\rangle$ cannot be distinguished by measurements: we write $|\phi\rangle \sim |\psi\rangle$.

Quantum states should really be thought of as *equivalence classes of vectors*

$$\{\,\alpha\,|\phi\rangle \;\mid\; \alpha \neq 0\,\}$$

i.e. lines in $\mathcal{V}$: elements of what the mathematicians call the **projective space** $\mathbb{P}^1(\mathcal{V})$.

Equivalence and normalization

Remark: clearly any quantum state is equivalent to a normalized state

$$|\phi\rangle \sim \frac{1}{\|\phi\|}|\phi\rangle$$

but such a normalized state is *not* unique:

$$|\phi\rangle \sim \alpha|\phi\rangle,$$

another state with the same norm, whenever $|\alpha| = 1$, *i.e.* $\alpha = e^{ia}$ $(a \in \mathbb{R})$

Quantum bits

Quantum systems

Dirac formalism

Quantum bits

Computational quantum systems

N-**level** quantum system: when $\dim_{\mathbb{C}} \mathcal{V} = N$.

Basis of pure (eigen) states $|\phi_0\rangle, |\phi_1\rangle, \ldots, |\phi_{N-1}\rangle$.

Computational basis : to simplify notation let us write

$$|n\rangle := |\phi_n\rangle \qquad (0 \leq n < N)$$

and $\mathcal{V}_N$ for the standard N-level state space with pure states

$$|0\rangle, |1\rangle, \ldots, |N-1\rangle.$$

$N = 1$ case: $|\phi\rangle = \alpha\,|0\rangle \sim |0\rangle$ "constant system" that behaves classically

$N = 2$: Quantum bits (or qubits)

The state of a qubit can be thought of as a nonzero linear combination

$$|\phi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \qquad \alpha, \beta \in \mathbb{C}.$$

When we measure it:

$$\mathbb{P}\big[\,\mathcal{M}|\phi\rangle = |0\rangle\,\big] = \frac{|\alpha|^2}{|\alpha|^2 + |\beta|^2}, \qquad \mathbb{P}\big[\,\mathcal{M}|\phi\rangle = |1\rangle\,\big] = \frac{|\beta|^2}{|\alpha|^2 + |\beta|^2}.$$

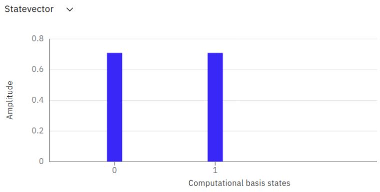
For a normalized state, $|\alpha|^2 + |\beta|^2 = 1$ so this is just

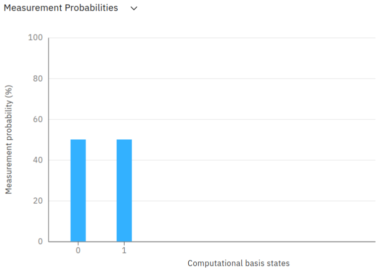
$$\mathbb{P}\big[\,\mathcal{M}|\phi\rangle = |0\rangle\,\big] = |\alpha|^2, \qquad \mathbb{P}\big[\,\mathcal{M}|\phi\rangle = |1\rangle\,\big] = |\beta|^2.$$

Example

$$|\phi\rangle = |0\rangle + |1\rangle \quad \sim \quad \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

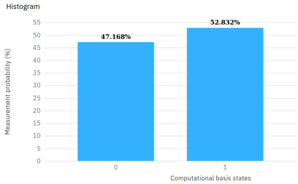
$$\mathbb{P}\left[\,\mathcal{M}|\phi\rangle = |0\rangle\,\right] = \mathbb{P}\left[\,\mathcal{M}|\phi\rangle = |1\rangle\,\right] = \frac{1}{2}$$



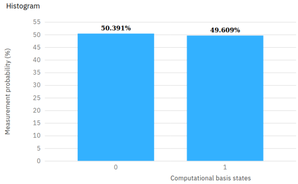


IBM Q Experience results

Result of 1024 *simulations*:



Result of 1024 *executions* on `ibmqx2`:



Your turn

Now would be a good time to create an account and start messing around with the

IBM Q Experience

https://quantum-computing.ibm.com/

Suggestion:

yields $$|\phi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$